

GammaLib Maths

Mathematical Implementation

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2.1.1 Small angle approximation

The shell function is a radial function $f(\theta)$, where θ is the angular separation between shell centre and the actual location. In the small angle approximation $\sin \theta \approx \theta$, the shell function is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\theta_{\text{out}}^2 - \theta^2} - \sqrt{\theta_{\text{in}}^2 - \theta^2} & \text{if } \theta \leq \theta_{\text{in}} \\ \sqrt{\theta_{\text{out}}^2 - \theta^2} & \text{if } \theta_{\text{in}} < \theta \leq \theta_{\text{out}} \\ 0 & \text{if } \theta > \theta_{\text{out}} \end{cases} \quad (1)$$

is the radial function. f_0 is a normalization constant that is determined by

$$2\pi \int_0^{\pi/2} f(\theta) \theta \, d\theta = 1 \quad (2)$$

in the small angle approximation. Using

$$\int x \sqrt{a - x^2} \, dx = -\frac{1}{3}(a - x^2)^{3/2} \quad (3)$$

we obtain

$$\frac{1}{f_0} = -\frac{2\pi}{3} \left[(\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{in}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} + (\theta_{\text{out}}^2 - \theta_{\text{out}}^2)^{3/2} - (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} \right] \quad (4)$$

$$= -\frac{2\pi}{3} \left[(\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} - (\theta_{\text{out}}^2 - \theta_{\text{in}}^2)^{3/2} \right] \quad (5)$$

$$= -\frac{2\pi}{3} \left[-(\theta_{\text{out}}^2)^{3/2} + (\theta_{\text{in}}^2)^{3/2} \right] \quad (6)$$

$$= \frac{2\pi}{3} [\theta_{\text{out}}^3 - \theta_{\text{in}}^3] \quad (7)$$

2.1.2 Spherical formulation

The shell function on a sphere is given by

$$f(\theta) = f_0 \begin{cases} \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} - \sqrt{\sin^2 \theta_{\text{in}} - \sin^2 \theta} & \text{if } \theta \leq \theta_{\text{in}} \\ \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} & \text{if } \theta_{\text{in}} < \theta \leq \theta_{\text{out}} \\ 0 & \text{if } \theta > \theta_{\text{out}} \end{cases} \quad (8)$$

The normalization constant f_0 is determined by

$$2\pi \int_0^{\pi/2} f(\theta) \sin \theta \, d\theta = 1 \quad (9)$$

Using

$$\int \sin x \sqrt{a - \sin^2 x} dx = -\frac{\cos x \sqrt{\cos 2x + 2a - 1}}{2\sqrt{2}} - \frac{a-1}{2} \ln \left(\sqrt{2} \cos x + \sqrt{\cos 2x + 2a - 1} \right) \quad (10)$$

and

$$2a = 2 \sin^2 \theta_0 = 1 - \cos 2\theta_0 \quad (11)$$

we can write

$$\int \sin x \sqrt{\sin^2 \theta_0 - \sin^2 x} dx = -\frac{\cos x \sqrt{\cos 2x - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln \left(\sqrt{2} \cos x + \sqrt{\cos 2x - \cos 2\theta_0} \right) \quad (12)$$

For the special case of $x = 0$, the integral becomes

$$-\frac{\sqrt{1 - \cos 2\theta_0}}{2\sqrt{2}} + \frac{\cos 2\theta_0 + 1}{4} \ln \left(\sqrt{2} + \sqrt{1 - \cos 2\theta_0} \right) \quad (13)$$

while for the special case $x = \theta_0$, it reduces to

$$\frac{\cos 2\theta_0 + 1}{4} \ln \left(\sqrt{2} \cos \theta_0 \right) \quad (14)$$

Using these equations, we can compute the required integrals:

$$\begin{aligned} I_1 = \int_0^{\theta_{\text{in}}} \sin \theta \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} d\theta &= \frac{\sqrt{1 - \cos 2\theta_{\text{out}}} - \cos \theta_{\text{in}} \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left(\frac{\sqrt{2} \cos \theta_{\text{in}} + \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{out}}}} \right) \end{aligned} \quad (15)$$

$$I_2 = \int_0^{\theta_{\text{in}}} \sin \theta \sqrt{\sin^2 \theta_{\text{in}} - \sin^2 \theta} d\theta = \frac{\sqrt{1 - \cos 2\theta_{\text{in}}}}{2\sqrt{2}} + \frac{\cos 2\theta_{\text{in}} + 1}{4} \ln \left(\frac{\sqrt{2} \cos \theta_{\text{in}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{in}}}} \right) \quad (16)$$

$$\begin{aligned} I_3 = \int_{\theta_{\text{in}}}^{\theta_{\text{out}}} \sin \theta \sqrt{\sin^2 \theta_{\text{out}} - \sin^2 \theta} d\theta &= \frac{\cos \theta_{\text{in}} \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left(\frac{\sqrt{2} \cos \theta_{\text{out}}}{\sqrt{2} \cos \theta_{\text{in}} + \sqrt{\cos 2\theta_{\text{in}} - \cos 2\theta_{\text{out}}}} \right) \end{aligned} \quad (17)$$

We thus obtain

$$\begin{aligned} \frac{1}{2\pi f_0} &= I_1 - I_2 + I_3 \\ &= \frac{\sqrt{1 - \cos 2\theta_{\text{out}}} - \sqrt{1 - \cos 2\theta_{\text{in}}}}{2\sqrt{2}} \\ &+ \frac{\cos 2\theta_{\text{out}} + 1}{4} \ln \left(\frac{\sqrt{2} \cos \theta_{\text{out}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{out}}}} \right) - \frac{\cos 2\theta_{\text{in}} + 1}{4} \ln \left(\frac{\sqrt{2} \cos \theta_{\text{in}}}{\sqrt{2} + \sqrt{1 - \cos 2\theta_{\text{in}}}} \right) \end{aligned} \quad (18)$$

The ratio in $\frac{1}{f_0}$ between the small angle approximation and the correct spherical computation is shown in Fig. ?? as function of θ_{out} for $\theta_{\text{in}} = \frac{2}{3}\theta_{\text{out}}$.

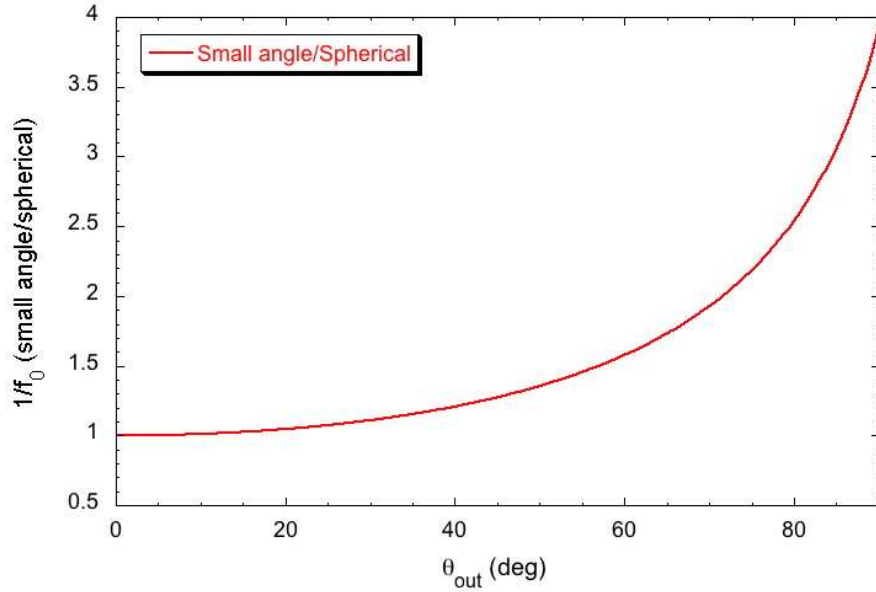


Figure 1: Ratio in $\frac{1}{f_0}$ between the small angle approximation and the correct spherical computation as function of θ_{out} for $\theta_{\text{in}} = \frac{2}{3}\theta_{\text{out}}$.

3 Spectral models

3.1 PowerLaw

The power law spectral model is defined by

$$I(E) = k \left(\frac{E}{p} \right)^\gamma \quad (19)$$

where

- k is the normalization of the power law (units: $\text{ph cm}^{-2} \text{ s}^{-1} \text{ MeV}^{-1}$),
- p is the pivot energy (units: MeV), and
- γ is the spectral index (which is usually negative).

Each of the 3 parameters is factorised into a scaling factor and a value, i.e. $k = k_s k_v$, $p = p_s p_v$, and $\gamma = \gamma_s \gamma_v$. The `GModelSpectralPlaw::eval_gradients` returns the gradients with respect to the parameter value of the factorisation. Note that for any parameter $a = a_s a_v$:

$$\frac{\delta I}{\delta a_v} = \frac{\delta I}{\delta a} \frac{\delta a}{\delta a_v} = \frac{\delta I}{\delta a} a_s \quad (20)$$

The parameter value gradients for the power law are given by

$$\frac{\delta I}{\delta k_v} = k_s \left(\frac{E}{p} \right)^\gamma = \frac{I(E)}{k_v} \quad (21)$$

$$\frac{\delta I}{\delta p_v} = -\frac{\gamma}{p_v} k \left(\frac{E}{p} \right)^\gamma = -\frac{\gamma}{p_v} I(E) \quad (22)$$

$$\frac{\delta I}{\delta \gamma_v} = \gamma_s \ln \left(\frac{E}{p} \right) k \left(\frac{E}{p} \right)^\gamma = \gamma_s \ln \left(\frac{E}{p} \right) I(E) \quad (23)$$

3.2 PowerLaw2

This flavour of the power law spectral model uses the integral flux f within the energy range E_{\min} and E_{\max} as free parameter instead of the normalization k . The integral flux f is given by

$$f = \int_{E_{\min}}^{E_{\max}} I(E) dE \quad (24)$$

The power law model is then defined by

$$I(E) = \tilde{k} E^{\gamma} \quad (25)$$

where

$$\tilde{k} = \begin{cases} \frac{f}{\ln E_{\max} - \ln E_{\min}} & \text{if } \gamma = -1 \\ \frac{f(1 + \gamma)}{E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1}} & \text{else} \end{cases} \quad (26)$$

is obtained from Eq. (??).

Each of the 4 parameters is factorised into a scaling factor and a value, e.g. $f = f_s f_v$ and $\gamma = \gamma_s \gamma_v$. It is assumed that E_{\min} and E_{\max} are fixed parameters, and `GModelSpectralPlaw2::eval_gradients` returns valid gradients only for f_v and γ_v .

The flux value gradient f_v is given by

$$\frac{\delta I}{\delta f_v} = \frac{\delta I}{\delta f} \frac{\delta f}{\delta f_v} = \frac{\delta I}{\delta f} f_s = \frac{\delta \tilde{k}}{\delta f} E^{\gamma} f_s = \frac{\tilde{k}}{f} E^{\gamma} f_s = \frac{I(E)}{f} f_s = \frac{I(E)}{f_v} \quad (27)$$

The index value gradient γ_v is given by

$$\frac{\delta I}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma} \frac{\delta \gamma}{\delta \gamma_v} = \frac{\delta I}{\delta \gamma} \gamma_s = \left(\frac{\delta \tilde{k}}{\delta \gamma} E^{\gamma} + \tilde{k} E^{\gamma} \ln E \right) \gamma_s = \left(\frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} + \ln E \right) \tilde{k} E^{\gamma} \gamma_s = \left(\frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} + \ln E \right) I(E) \gamma_s \quad (28)$$

where

$$\frac{\delta \tilde{k}}{\delta \gamma} = \begin{cases} 0 & \text{if } \gamma = -1 \\ \frac{f \left(E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right) - f(1 + \gamma) \left(E_{\max}^{\gamma+1} \ln E_{\max} - E_{\min}^{\gamma+1} \ln E_{\min} \right)}{\left(E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right)^2} & \text{else} \end{cases} \quad (29)$$

Note that for $\gamma \neq -1$

$$\frac{1}{\tilde{k}} \frac{\delta \tilde{k}}{\delta \gamma} = \frac{1}{1 + \gamma} - \frac{\left(E_{\max}^{\gamma+1} \ln E_{\max} - E_{\min}^{\gamma+1} \ln E_{\min} \right)}{\left(E_{\max}^{\gamma+1} - E_{\min}^{\gamma+1} \right)} \quad (30)$$